ON FIXED POINTS OF HAMMERSTEIN OPERATOR WITH DEGENERATE KERNEL

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We construct a degenerate kernel such that Hammerstein operator has exactly three positive fixed points.

Let $\varphi_1(u), \varphi_2(u)$ and $\psi_1(t), \psi_2(t)$ be positive functions from from $C_0{}^+[0,\ 1] := \{f\ \in C[0,1 : f(x) \geq 0]\} \setminus \{f(x) \neq \theta\}$. We consider special type of Hammerstein operator H with degenerate kernel (see [1]):

$$(Hf\ )(t) = \int_0^1 \left(\varphi_1(u)\psi_1(t) + \ \varphi_2(u)\psi_2(t)\right) f^2(u)du.$$

We consider symmetrical matrix 2 × 2 $A = (a_{ij})_{i,j=1,2}$ and $B = (b_{ij})_{i,j=1,2}$ with (strictly) positive elements. For matrix A and B we define quadratic
operator (QO) Q in cone of the space R^2 by the rule:

$$Q(x,y) = (\alpha_{11}x^2 + 2\alpha_{12}xy + \ \alpha_{22}y^2\ ,\ \ \beta_{11}x^2 + 2\beta_{12}xy + \beta_{22}y^2).$$

Denote

$$\alpha_{11} = \int_0^1 \psi_1(t)\varphi_1{}^2(u)du\ > 0, \qquad \alpha_{12} = \int_0^1 \psi_1(t)\varphi_1(u)\varphi_2(u)du\ > 0,$$

$$\alpha_{22} = \int_0^1 \psi_1(t)\varphi_2{}^2(u)\mathrm{du} > 0;$$

$$\beta_{11} = \int_0^1 \psi_2(t)\varphi_1{}^2(u)du\ > 0, \qquad \beta_{12} = \int_0^1 \psi_2(t)\varphi_1(u)\varphi_2(u)du\ > 0,$$

$$\beta_{22} = \int_0^1 \psi_2(t)\varphi_2{}^2(u)\mathrm{du} > 0.$$

Clearly, an arbitrary nontrivial positive fixed points of the QO Q is strictly positive (see [2]).

Lemma [**3**]. Following statements hold**:**

(1) If the point $\omega = (x_0, y_0)$ is a strictly fixed point of QO Q, then $\xi_0 = \frac{y_0}{x_0}$ is a root of the cubic algebraic equation

$$\alpha_{22}\xi^3 + (2\alpha_{12} - \beta_{22})\xi^2 + (\alpha_{11} - 2\beta_{12})\xi - \beta_{11} = 0.$$

(2) If the point $\omega = (x_0, y_0) \in R_2{}^+$ is a fixed point of QO Q, then $\omega \in R_2{}^>$ and $\xi_0 = \frac{y_0}{x_0}$ is a root of the cubic algebraic equation

$$\alpha_{22}\xi^3 + (2\alpha_{12} - \beta_{22})\xi^2 + (\alpha_{11} - 2\beta_{12})\xi - \beta_{11} = 0.$$

We define two continuous positive functions $\varphi_1(u)$ and $\varphi_2(u)$ on [0, 1]

$$\varphi_1(u) = \begin{cases} u + \frac{1}{16}\,, & if\ u \in \left[0, \frac{1}{4}\right] \\ \frac{9}{16} - u\,, & if\ u \in \left[\frac{1}{4}, \frac{1}{2}\right] \\ \frac{1}{16}\,, & if\ u \in \left[\frac{1}{2}, 1\right] \end{cases}$$

and

$$\varphi_2(u)=\begin{cases}\frac{1}{16}, & if\ u\in\left[0,\frac{1}{2}\right]\\ u-\frac{7}{16}, & if\ u\in\left[\frac{1}{2},\frac{3}{4}\right]\\ \frac{17}{16}-u, & if\ u\in\left[\frac{3}{4},1\right]\end{cases}$$

We define two continuous positive functions $\psi_1(t)$ and $\psi_2(t)$ on [0, 1]:

$$\psi_1(t)=\begin{cases}260-512t, & if\ t\in\left[0,\frac{1}{2}\right],\\ 4, & if\ t\in\left[\frac{1}{2},1\right].\end{cases}$$

$$\psi_2(t)=\begin{cases}4, & if\ t\in\left[0,\frac{1}{2}\right]\\ 512u-252, & if\ t\in\left[\frac{1}{2},1\right]\end{cases}$$

Denote

$$K(t,u)=\varphi_1(u)\psi_1(t)+\varphi_2(u)\psi_2(t),\ \ t,u\in[0,1].$$

Theorem. Let $a,b\in R^+$. Then there exist exactly three positive fixed points of the operator H_2 with the kernel $K\ (t,u)$.

Reference

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[3]. Yu. Kh. Eshkabilov, Sh. D. Nodirov, F. H. Haydarov: Positive fixed points of quadratic operators and Gibbs Measures, Positivity 20(4) (2016).